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IFAC-PapersOnLine 48-25 (2015) 254–258

On Guarantee Optimization under Conditions of Integral Constraints on Control Actions and Nonterminal Quality Index

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Abstract: A linear-convex problem of dynamical guarantee optimization under conditions of unknown disturbances and with positional quality index that estimates a set of deviations of the motion of the controlled system at given instants of time from given target points is considered. Control actions are bounded by both geometrical and integral constraints. Disturbance actions are only geometrically bounded. A procedure for approximate computing of the optimal guaranteed result and of the corresponding optimal closed-loop control law is elaborated. The method is based on recurrent constructions of upper convex hulls of auxiliary program functions. Results of numerical experiments on model examples are given.

Keywords: Optimal control, unknown disturbances, differential games, optimal guaranteed result, optimal control strategies.

1. INTRODUCTION

Within the game-theoretic approach (see, e.g., Krasovskii (1985); Krasovskii and Krasovskii (1995)) for a linear dynamical system a control problem under conditions of unknown disturbances is considered. Control actions are bounded by norm by both geometrical and impulse-integral constraints. Disturbance actions are bounded by geometrical constraints only. Due to geometrical constraints impulse problem statement (see, e.g., Krasovskii and Tretyakov (1966); Nikolskii (1972); Subbotina and Subbotin (1975)) and difficulties connected with it do not arise here. Still the presence of integral constraints leads to additional complications connected with optimizations of resource expenditure by the control. Another distinctive feature of the considered problem lies in the nonterminal quality index (payoff functional) that evaluates a set of deviations of the motion at given instants of time from given target points. In this paper a procedure for computing the optimal guaranteed result of the control is elaborated. It is based on recurrent constructions of upper convex hulls of appropriate auxiliary functions. On the basis of this procedure and the method of extremal shift to the accompanying point (see, e.g., Krasovskii (1985); Krasovskii and Krasovskii (1995)) such a closed loop control law is constructed that within a given ahead precision ensures a result not worse than the optimal guaranteed result. The applicability of the developed procedure is verified by computer simulations on model examples.

The method of upper convex hulls, which we extend in this paper, conceptually goes back to stochastic program synthesis (see Krasovskii (1985)). Originally this method was developed for problems without integral constraints (see, e.g., Krasovskii and Krasovskii (1995); Krasovskii (1987); Krasovskii and Reshetova (1988); Krasovskii and Lukoyanov (1996); Lukoyanov (1998, 2001)). Its modifications for problems with integral constraints were considered by Lokshin (1992); Lukoyanov (1995a,b), but only for the case of terminal quality indices. In this paper we unite the constructions from Lukoyanov (1998) and Lukoyanov (1995a,b) in order to solve control problems with both integral constraints and nonterminal quality indices.

⋆ This work is supported by the Russian Foundation for Basic Research (11-01-12088-ofi-m-2011), by the Program for State Support of Leading Scientific Schools of Russia (NSh-5927.2012.1), and by the Program of the Presidium of the Russian Academy of Sciences "Fundamental Problems of Nonlinear Dynamics in Mathematics and Physics" (12-P-1-1012).

2. PROBLEM STATEMENT

Consider a dynamical system described by the following equation

$$\begin{aligned} \dot{x} = A(t)x + B(t)u + C(t)v, \quad t_0 \le t < \vartheta, \\ x \in \mathbb{R}^n, \quad u \in \mathbb{R}^{n_u}, \quad v \in \mathbb{R}^{n_v}, \end{aligned} \tag{1}$$

Here x is a phase vector; t is time; $\dot{x}(t) = \mathrm{d}x(t)/\mathrm{d}t$; matrix functions $A(t)$, $B(t)$, and $C(t)$ are continuous; u is a control vector, v is a disturbance vector; t_0 and ϑ are fixed instants of time.

Any Borel-measurable realization $u(\cdot) = \{u(t) \in \mathbb{R}^{n_u},\ t_0 \le t < \vartheta\}$ of control is admissible, if it satisfies the following geometrical and integral constraints

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2015.11.096

$$\|u(t)\|_u \le M_u, \quad t_0 \le t < \vartheta; \qquad \int_{t_0}^{\vartheta} \alpha(\tau)\|u(\tau)\|_u d\tau \le \rho_0. \tag{2}$$

Realization $v(\cdot) = \{v(t) \in \mathbb{R}^{n_v},\ t_0 \le t < \vartheta\}$ of disturbance is admissible, if it is Borel-measurable and satisfies only the geometrical constraint

$$\|v(t)\|_v \le M_v, \quad t_0 \le t < \vartheta. \tag{3}$$

Here $\|\cdot\|_u$ and $\|\cdot\|_v$ are norms in $\mathbb{R}^{n_u}$ and $\mathbb{R}^{n_v}$; M_u, M_v, ρ_0 are given positive numbers; $\alpha(\tau)$ is a function that is scalar, positive and continuous on $[t_0, \vartheta]$.

The system motion generated by admissible realizations $u(\cdot)$ and $v(\cdot)$ is an absolutely continuous function $\{x(t) \in \mathbb{R}^n,\ t_0 \le t \le \vartheta\}$ that together with $u = u(t)$ and $v = v(t)$ for almost every t satisfies equation (1) and initial condition

$$x(t_0) = x_0 \in \mathbb{R}^n. \tag{4}$$

We consider a problem in which the goal of control u is to minimize the quality index

$$\gamma\big(x(\cdot)\big) = \mu_1\Big(D_1\big(x(\vartheta_1) - c_1\big), \ldots, D_N\big(x(\vartheta_N) - c_N\big)\Big), \tag{5}$$

where $\vartheta_i \in (t_0, \vartheta]$: $\vartheta_{i+1} > \vartheta_i,\ i = 1, \ldots, N-1,\ \vartheta_N = \vartheta$, are given instants of motion quality evalutaion; D_i are constant matrices with dimensions $d_i \times n$ $(1 \le d_i \le n)$; $c_i \in \mathbb{R}^n$ are target vectors; $\mu_1(l_1, \ldots, l_N)$ is a norm in space of $(d_1 + \ldots + d_N)$-dimensional vector-collections $(l_1, \ldots, l_N)$, composed of d_i-dimensional vectors $l_i,\ i = 1, \ldots, N$. We assume that there exist such norms $\mu_i\big(l_i, \ldots, l_N\big)$ and $\sigma_i(l_i, \mu)$ that the following equalities hold for $i = 1, \ldots, N-1$

$$\mu_i(l_i, \ldots, l_N) = \sigma_i(l_i, \mu), \quad \mu = \mu_{i+1}(l_{i+1}, \ldots, l_N). \tag{6}$$

Then (see Lukoyanov (1998)) quality index γ is positional (Krasovskii and Krasovskii, 1995, p. 43). Common examples of indices that satisfy conditions (5), (6) are

$$\gamma_1(x(\cdot)) = \sum_{i=1}^{N} \|x(\vartheta_i) - c_i\|,$$

$$\gamma_2(x(\cdot)) = \left(\sum_{i=1}^{N} \|x(\vartheta_i) - c_i\|^2\right)^{\frac{1}{2}},$$

$$\gamma_\infty(x(\cdot)) = \max_{i=1,\ldots,N} \|x(\vartheta_i) - c_i\|,$$

where symbol $\|\cdot\|$ denotes some norm in $\mathbb{R}^n$.

Since disturbance actions are unknown, in particular, they may be aimed at maximization of quality index (5).

In addition to phase vector x of system (1) let us introduce variable ρ whose evolution is described by equation

$$\dot\rho = -\alpha(t)\|u\|_u, \quad t_0 \le t < \vartheta. \tag{7}$$

Then integral constraint from (2) can be rewritten in the form of phase constraints on ρ: $0 \le \rho \le \rho_0$.

Control strategy U is a vector-function

$$U = U(t, x, \rho, \varepsilon) \in \mathbb{R}^{n_u},\ \|U(t, x, \rho, \varepsilon)\|_u \le M_u,$$
$$t_0 \le t < \vartheta,\ x \in \mathbb{R}^n,\ 0 \le \rho \le \rho_0,\ \varepsilon > 0.$$

Here ε is a precision parameter (Krasovskii, 1985, p. 68), whose value is chosen before the beginning of the control process, stays unchanged during this process and defines the precision of the problem solution.

Control law $\mathcal{U}$ is a triple $\{U,\ \varepsilon,\ \Delta_\delta\}$, where Δ_δ is a partition of the control interval $[t_0, \vartheta]$:

$$\begin{aligned}\Delta_\delta = \{t_j\colon t_1 = t_0,\ 0 < t_{j+1} - t_j \le \delta,\\ j = 1, \ldots, k,\ t_{k+1} = \vartheta\}.\end{aligned} \tag{8}$$

From a given position $\{t_0, x_0, \rho_0\}$ the control law $\mathcal{U}$ together with an admissible realization $v(\cdot)$ of disturbance unambiguously forms the motion $\{x(\cdot), \rho(\cdot)\}$ of extended system (1), (7) that is defined as a solution of the following step-by-step equations

$$\begin{aligned}\dot x(t) &= A(t)x(t) + B(t)u_j(t) + C(t)v(t),\\ \dot\rho(t) &= -\alpha(t)\|u_j(t)\|_u,\\ &t_j \le t < t_{j+1},\ j = 1, \ldots, k,\end{aligned} \tag{9}$$

with the initial condition

$$x(t_1) = x_0, \quad \rho(t_1) = \rho_0.$$

Here, for $t_j \le t < t_{j+1}$,

$$u_j(t) = \begin{cases} 0, & \text{if } 0 \le \rho(t_j) < \displaystyle\int_{t_j}^{t_{j+1}} \alpha(\tau)\|u_j^*\|_u d\tau,\\ u_j^*, & \text{if } \rho(t_j) \ge \displaystyle\int_{t_j}^{t_{j+1}} \alpha(\tau)\|u_j^*\|_u d\tau,\end{cases} \tag{10}$$

where $u_j^* = U\big(t_j, x(t_j), \rho(t_j), \varepsilon\big)$.

The guaranteed result of control law $\mathcal{U}$ for a fixed initial position $\{t_0, x_0, \rho_0\}$ is

$$\Gamma(U; t_0, x_0, \rho_0) = \limsup_{\varepsilon\to 0} \lim_{\delta\to 0} \sup_{\Delta_\delta} \sup_{v(\cdot)} \gamma.$$

Here Δ_δ is partition (8), supremum is taken over all admissible realizations $v(\cdot)$ (3), γ is a value of index (5) that was realized on motion $x(\cdot)$, formed in accordance with (9), (10) by the control law $\mathcal{U} = \{U, \varepsilon, \Delta_\delta\}$ together with the realization $v(\cdot)$ of disturbance from the initial position $\{t_0, x_0, \rho_0\}$.

The optimal guaranteed result $\Gamma^0(t_0, x_0, \rho_0)$ of control is

$$\Gamma^0(t_0, x_0, \rho_0) = \inf_U \Gamma(U; t_0, x_0, \rho_0).$$

In compliance with this definition, for $\zeta > 0$, a control law $\mathcal{U}$ is ζ-optimal, if according to (9), (10) it guarantees inequality

$$\gamma \le \Gamma^0(t_0, x_0, \rho_0) + \zeta, \tag{11}$$

for any admissible realization $v(\cdot)$ of disturbance.

The aim of this paper is to elaborate a procedure for computation of the optimal guaranteed result $\Gamma^0(t_0, x_0, \rho_0)$ of control and of the ζ-optimal control laws.

3. OPTIMAL GUARANTEED RESULT

Fix a partition Δ_δ like (8). Include in it all instants ϑ_i of motion quality evaluation from index (5):

$$\vartheta_i \in \Delta_\delta, \quad i = 1, \ldots, N. \tag{12}$$

Let $j = 1, \ldots, k+1$, $\rho \ge 0$, $m \in \mathbb{R}^n$. By $W_j^u(\rho)$ denote such a set of measurable functions $u(\cdot) = \{u(t) \in \mathbb{R}^{n_u},\ t_j \le t < t_{j+1}\}$ that satisfy the following conditions:

$$\|u(t)\|_u \le M_u,\ t_j \le t < t_{j+1}; \qquad \int\limits_{t_j}^{t_{j+1}} \alpha(\tau)\|u[\tau]\|_u d\tau \le \rho.$$

By W_j^v denote a set of all measurable functions $v(\cdot) = \{v(t) \in \mathbb{R}^{n_v},\ t_j \le t < t_{j+1}\}$ such that

$$\|v(t)\|_v \le M_v,\ t_j \le t < t_{j+1}.$$

Set

$$\Delta\psi_j(m,\rho) = \min_{u(\cdot)\in W_j^u(\rho)} \int\limits_{\tau_j}^{\tau_{j+1}} \langle m, X[\vartheta,\tau]B(\tau)u(\tau)\rangle d\tau +$$
$$+ \max_{v(\cdot)\in W_j^v} \int\limits_{\tau_j}^{\tau_{j+1}} \langle m, X[\vartheta,\tau]C(\tau)v(\tau)\rangle d\tau.$$

Here $X[\vartheta,\tau]$ is a fundamental solution matrix of the equation $\dot{x}(t) = A(t)x(t)$ such that $X[\tau,\tau] = E$, where E is the identity matrix. Symbol $\langle\cdot,\cdot\rangle$ denotes the inner product of vectors. It can be verified that the sets $W_j^u(\rho)$ and W_j^v are weakly compact in $L_2(t_j, t_{j+1})$ space of square integrable functions, functions $\Delta\psi_j(m,\rho)$ are continuous with respect to (m,ρ) and are convex and nonincreasing with respect to ρ.

Denote

$$h(t) = \min_{i=1,\ldots,N}\{i : \vartheta_i \ge t\},\ t_0 \le t \le \vartheta.$$

Step by step, in the reverse order, starting from the last point of the partition Δ_δ, define sets $G_j^\pm$ of vectors $m \in \mathbb{R}^n$ and scalar functions $\varphi_j^\pm(m,\rho)$, $m \in G_j^\pm$, $\rho \ge 0$, $j = 1,\ldots,k+1$, by the following recurrent relations.

For $j = k+1$, set

$$G_{k+1}^+ = \{m : m = 0\},$$
$$\varphi_{k+1}^+(m,\rho) = 0,\ m \in G_{k+1}^+,\ \rho \ge 0,$$
$$G_{k+1}^- = \{m : m = D_N^\top l,\ l \in \mathbb{R}^{d_N},\ \mu_N^*(l) \le 1\},$$
$$\varphi_{k+1}^-(m,\rho) = -\langle m, c_N\rangle,\ m \in G_{k+1}^-,\ \rho \ge 0.$$

For $1 \le j \le k$, set

$$G_j^+ = G_{j+1}^-,$$
$$\varphi_j^+(m,\rho) = \{\psi_j(\cdot,\rho)\}^*_{G_j^+}(m), \quad m \in G_j^+,\ \rho \ge 0,$$

where

$$\psi_j(m,\rho) = \min_{\rho'\in R_j(\rho)}[\Delta\psi_j(m,\rho-\rho') + \varphi_{j+1}^-(m,\rho')],$$
$$m \in G_j^+,\ \rho \ge 0,$$
$$R_j(\rho) = \{\rho' : \max[0, \rho - M_u \int\limits_{t_j}^{t_{j+1}} \alpha(\tau)d\tau] \le \rho' \le \rho\},$$

and then, if t_j is not equal to any instant ϑ_i of motion quality evaluation, i.e. $t_j < \vartheta_{h(t_j)}$, set

$$G_j^- = G_j^+,$$
$$\varphi_j^-(m,\rho) = \varphi_j^+(m,\rho),\ m \in G_j^-,\ \rho \ge 0,$$

otherwise, when $t_j = \vartheta_h$, $h = h(t_j)$, set

$$G_j^- = \{m : m = \nu m_* + X^\top[\vartheta_h,\vartheta]D_h^\top l,$$
$$\nu \ge 0, l \in \mathbb{R}^{d_h}, \sigma_h^*(l,\nu) \le 1, m_* \in G_j^+\},$$
$$\varphi_j^-(m,\rho) = \max_{\{\nu,m_*,l\}|m}[\nu\varphi_j^+(m_*,\rho) - \langle l, D_h c_h\rangle],$$
$$m \in G_j^-,\ \rho \ge 0.$$

Here superscript "⊤" denotes the matrix transposition; $\mu_N^*(\cdot)$ and $\sigma_h^*(\cdot)$ are norms dual to $\mu_N(\cdot)$ and $\sigma_h(\cdot)$ from (6); for every fixed $\rho \ge 0$, symbol $\{\psi_j(\cdot,\rho)\}^*_{G_j^+}$ denotes the upper convex hull of the function $\psi_j(\cdot,\rho)$ on the set G_j^+, i.e. the minimal concave function that majorizes $\psi_j(m,\rho)$ for $m \in G_j^+$; maximum for $\varphi_j^-(m,\rho)$ is calculated over all such triples $\{\nu, m_*, l\}$, $m_* \in G_j^+$, $\nu \ge 0$, $l \in \mathbb{R}^{d_h}$, $\sigma_h^*(l,\nu) \le 1$, that satisfy the equality $\nu m_* + X^\top[\vartheta_h,\vartheta]D_h^\top l = m$.

It can be proved that for every $j = 1,\ldots,k+1$ the above constructed sets $G_j^\pm$ are convex compacts in $\mathbb{R}^n$ that contain vector $m = 0$. If unit balls of norms $\mu_i^*(\cdot)$, $i = 1,\ldots,N$ are strictly convex or are polyhedra or more generally are P-sets (Balashov (2002)), then functions $\varphi_j^\pm(m,\rho)$ are continuous with respect to m for fixed $\rho \ge 0$ (Gomoyunov and Lukoyanov (2015)). For fixed $m \in G_j^\pm$ functions $\varphi_j^\pm(m,\rho)$ are nonincreasing, continuous, and convex with respect to $\rho \ge 0$. Besides, $\varphi_j^\pm(0,\rho) \ge 0$.

For $j = 1,\ldots,k+1$, consider the system of values

$$e_j^\pm(x,\rho) = \max_{m\in G_j^\pm}[\langle m, X[\vartheta,t_j]x\rangle + \varphi_j^\pm(m,\rho)]. \tag{13}$$

Theorem 1. For any bounded subset $X_0 \subset \mathbb{R}^n$ and any number $\xi > 0$, there exists a number $\delta^* > 0$ such that for any partition Δ_δ, $\delta \le \delta^*$, like (8), (12), the following inequality holds

$$|\Gamma^u(t_0,x_0,\rho_0) - e_1^-(x_0,\rho_0)| \le \xi, \quad x_0 \in X_0, \quad \rho_0 \ge 0.$$

This theorem is a corollary of the following properties of system of values (13).

Lemma 2. For any partition Δ_δ like (8), (12), value $j = 1,\ldots,k$, and values $x \in \mathbb{R}^n$ and $\rho \ge 0$

$$e_j^-(x,\rho) = \begin{cases} e_j^+(x,\rho), & \text{if } t_j < \vartheta_h, \\ \sigma_h(D_h(x-c_h), e_j^+(x,\rho)), & \text{if } t_j = \vartheta_h, \end{cases}$$

where $h = h(t_j)$.

The proof of lemma 2 follows the scheme of the proof of analogous lemma from Lukoyanov (1998) that was formulated for the case when there are no integral constraints on control actions.

Lemma 3. (u-stability). For any $\varepsilon > 0$, there exists $\delta^* > 0$ such that, for any $j = 1,\ldots,k$, $x_* \in \mathbb{R}^n$, $\rho_* \ge 0$, and partition Δ_δ,$\delta \le \delta^*$, like (8), (12), for any realization $v(\cdot) \in W_j^v$ of disturbance, there exists such realization $u(\cdot) \in W_j^u(\rho_*)$ of control that the motion of extened system (1), (7), formed by these realizations from the position $\{t_j, x(t_j) = x_*, \rho(t_j) = \rho_*\}$, comes to such position $\{t_{j+1}, x^* = x(t_{j+1}), \rho^* = \rho(t_{j+1})\}$, for which the following inequality holds

$$e_{j+1}^-(x^*,\rho^*) - e_j^+(x_*,\rho_*) \le \varepsilon(t_{j+1} - t_j).$$

Lemma 4. (v-stability). For any $j = 1,\ldots,k$, $x_* \in \mathbb{R}^n$, $\rho_* \ge 0$, and partition Δ_δ like (8), (12), there exists such realization $v(\cdot) \in W_j^v$ of disturbance that, for any realization $u(\cdot) \in W_j^u(\rho_*)$ of control that the motion of extened system (1), (7), formed by these realizations from the position $\{t_j, x(t_j) = x_*, \rho(t_j) = \rho_*\}$, comes to such position $\{t_{j+1}, x^* = x(t_{j+1}), \rho^* = \rho(t_{j+1})\}$, for which the following inequality holds

$$e_{j+1}^-(x^*, \rho^*) - e_j^+(x_*, \rho_*) \geq 0.$$

The proofs of lemmas 3 and 4 follow the schemes of the proofs of analogous lemmas from Lukoyanov (1995a) (see also Lukoyanov (1995b)) that were formulated for problems with terminal quality indices.

4. ζ-OPTIMAL CONTROL LAW

For $\varepsilon > 0$, set

$$\varepsilon(t) = \sqrt{\varepsilon + \varepsilon(t - t_0)} \exp\{\lambda(t - t_0)\},\ t \in [t_0, \vartheta],$$
$$\lambda = \sup_{t \in [t_0, \vartheta],\, \|x\| = 1} \|A(t)x\|.$$

Here and below the symbol $\|\cdot\|$ denotes the Euclidean vector norm.

Define a control law $\mathcal{U}^{\mathrm{e}} = \{U^{\mathrm{e}}_{\Delta_\delta}, \varepsilon, \Delta_\delta\}$ that according to the scheme (9), (10) forms the value $u_j^* = U^{\mathrm{e}}_{\Delta_\delta}(t_j, x(t_j), \rho(t_j), \varepsilon)$ in the following way:

$$U^{\mathrm{e}}_{\Delta_\delta}(t_j, x, \rho, \varepsilon) = \begin{cases} 0, & \text{if } \rho < \varepsilon(t_j), \\ u_j^{\mathrm{e}}, & \text{otherwise,} \end{cases}$$

where u_j^{e} is found from the condition of the extremal shift to accompanying point (Krasovskii (1985); Lukoyanov (1995a)):

$$u_j^{\mathrm{e}} \in \operatorname*{arg\,min}_{\|u\|_u \leq M_u} \left[\langle s_j^u, B(t_j)u\rangle - r_j^u \alpha(t_j)\|u\|_u\right],$$

where

$$s_j^u = x - x_j^u, \quad r_j^u = \rho - \rho_j^u,$$
$$\{x_j^u, \rho_j^u\} \in \operatorname*{arg\,min}_{\|\{x,\rho\} - \{x_j,\rho_j\}\| \leq \varepsilon(t_j)} e_j^+(x, \rho).$$

Due to the definition (13) of the value $e_j^+(\cdot)$, the following equalities for values s_j^u, r_j^u can be obtained:

$$s_j^u = \frac{X^\top[\vartheta, t] m_j^u}{\sqrt{1 + \|X^\top[\vartheta, t] m_j^u\|^2}} \sqrt{\varepsilon^2(t) - (r_j^u)^2},\ r_j^u = r_j^u(m_j^u),$$

$$r_j^u(m) \in \operatorname*{arg\,max}_{|r| \leq \varepsilon(t)} \Big[\sqrt{1 + \|X^\top[\vartheta, t]m\|^2}\sqrt{\varepsilon^2(t) - r^2} - $$
$$- \varphi_j^+(m, \rho - r)\Big],$$

$$m_j^u \in \operatorname*{arg\,max}_{m \in G_j^+} \Big[\langle X^\top[\vartheta, t]m, x\rangle -$$
$$- \sqrt{1 + \|X^\top(\vartheta, t)m\|^2}\sqrt{\varepsilon^2(t) - (r_j^u(m))^2} +$$
$$+ \varphi_j^+(m, \rho - r_j^u(m))\Big].$$

For proof of similar equalities see, e.g., Krasovskii and Reshetova (1988); Kornev (2012).

Theorem 5. For any bounded set $X_0 \subset \mathbb{R}^n$ and any number $\zeta > 0$ there exist a number $\varepsilon^* > 0$ and a function $\delta(\varepsilon) > 0,\ 0 < \varepsilon \leq \varepsilon^*$, such that, for any $x_0 \in X_0$, $\rho_0 \geq 0$, value $0 < \varepsilon \leq \varepsilon^*$, and partition Δ_δ, $\delta \leq \delta(\varepsilon)$, control law $\mathcal{U}^{\mathrm{e}} = \{U^{\mathrm{e}}_{\Delta_\delta}, \varepsilon, \Delta_\delta\}$ guarantees the inequality

$$\gamma \leq \Gamma^0(t_0, x_0, \rho_0) + \zeta,$$

for any admissible realization $v(\cdot)$ of disturbance.

So, according to (11), the control law $\mathcal{U}^{\mathrm{e}}$ is ζ-optimal. The proof of the theorem 5 follows the scheme from (Krasovskii, 1985, p. 207–215) and relies on lemmas 2 and 3.

Note that if one considers problem (1)–(5) from the point of view of the disturbance aimed at maximizing the quality index, then in analogous way, taking into account that there are no integral constraints on the disturbance actions, a counter-optimal law $\mathcal{V}^{\mathrm{e}} = \{V^{\mathrm{e}}_{\Delta\delta}, \varepsilon, \Delta_\delta\}$ can be constructed, which forms the disturbance actions

$$v(t) = V^{\mathrm{e}}_{\Delta_\delta}(t_j, x(t_j), \rho(t_j), \varepsilon),$$
$$t_j \leq t < t_{j+1}, \quad j = 1, \ldots, k+1,$$

on the basis of the condition of extremal shift:

$$V^{\mathrm{e}}_{\Delta_\delta}(t_j, x, \rho, \varepsilon) = v_j^{\mathrm{e}},$$
$$v_j^{\mathrm{e}} \in \operatorname*{arg\,max}_{\|v\|_v \leq M_v} \left[\langle s_j^v, C(t_j)v\rangle\right], \quad s_j^v = x_j^v - x,$$
$$x_j^v \in \operatorname*{arg\,max}_{\|x - x_j\| \leq \varepsilon(t_j)} e_j^+(x, \rho),$$

where, taking into account (13), value s_j^v can be computed from formulas

$$s_j^v = \frac{X^\top[\vartheta, t_j] m_j^v}{\sqrt{1 + \|X^\top[\vartheta, t_j] m_j^v\|^2}} \varepsilon(t_j),$$

$$m_j^v \in \operatorname*{arg\,max}_{m \in G_j^+} \Big[\langle m, X[\vartheta, t_j]x_j\rangle +$$
$$+ \varepsilon(t_j)\sqrt{1 + \|X^\top[\vartheta, t_j]m\|^2} + \varphi_j^+(m, \rho)\Big].$$

5. EXAMPLE

The elaborated procedure for computing the optimal guaranteed result and the method for constructing the corresponding ζ-optimal control law were implemented in a piece of software. Details of the analogous software implementation are given in Kornev (2012).

By means of the developed software implementation the following results of computer simulations were obtained.

Consider a dynamical system described by the following equation

$$\begin{cases} \dot{x}_1(t) = x_2(t) + c(t)v(t), & t_0 = 0 \leq t < \vartheta = 2, \\ \dot{x}_2(t) = -0.5x_1(t) - 0.05x_2(t) + b(t)u(t), \end{cases}$$
$$x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2, \quad u \in \mathbb{R}^1, \quad v \in \mathbb{R}^1,$$
$$b(t) = \begin{cases} 2 + 2\cos 2\pi(t - 0.5), & \text{if } t \in [0.5; 1.5], \\ 4 & \text{otherwise,} \end{cases}$$
$$c(t) = \begin{cases} 0.3, & \text{if } t \in [0.6; 1.4], \\ 0.1 & \text{otherwise,} \end{cases}$$

constraints on control actions and disturbance actions are

$$\|u(t)\| \leq 1, \|v(t)\| \leq 1, \leq t < 2, \quad \int_0^2 \|u(\tau)\| d\tau \leq \rho_0 = 1,$$

initial condition

$$x(0) = (x_1(0), x_2(0)) = (0.5, 0.1)$$

and quality index

$$\gamma = \sqrt{|x_1(1)|^2 + |x_2(1) - 0.5|^2 + |x_1(2) + 0.5|^2 + |x_2(2)|^2}$$

are given.

The following values of the parameters were used in the numerical experiment: the diameter $\delta = 0.01$ of the partition Δ_δ of the control interval and the value of the precision parameter $\varepsilon = 0.05$. The a priori calculated value $e_1^-(x_0, \rho_0)$ that approximates the optimal guaranteed result of the control in the considered problem was equal to 0.702.

Three control simulations were performed.

In the first simulation actions of control and disturbance were formed by laws $\mathcal{U}^e$ and $\mathcal{V}^e$, respectively. Under these conditions the resulting value of the quality index (5) was

$$\gamma^{(1)} = \big(|0.235|^2 + |-0.093 - 0.5|^2 + |-0.184 + 0.5|^2 + \\ + |-0.170|^2\big)^{1/2} \approx 0.731 \approx e_1^-(x_0, \rho_0) = 0.702.$$

The motion realization obtained in this simulation is depicted in fig. 1. The targets are pointed out by black squares. White circles on the motion realization correspond to instants of time of motion quality evaluation. Realization of control actions together with function $b(t)$ is also given.

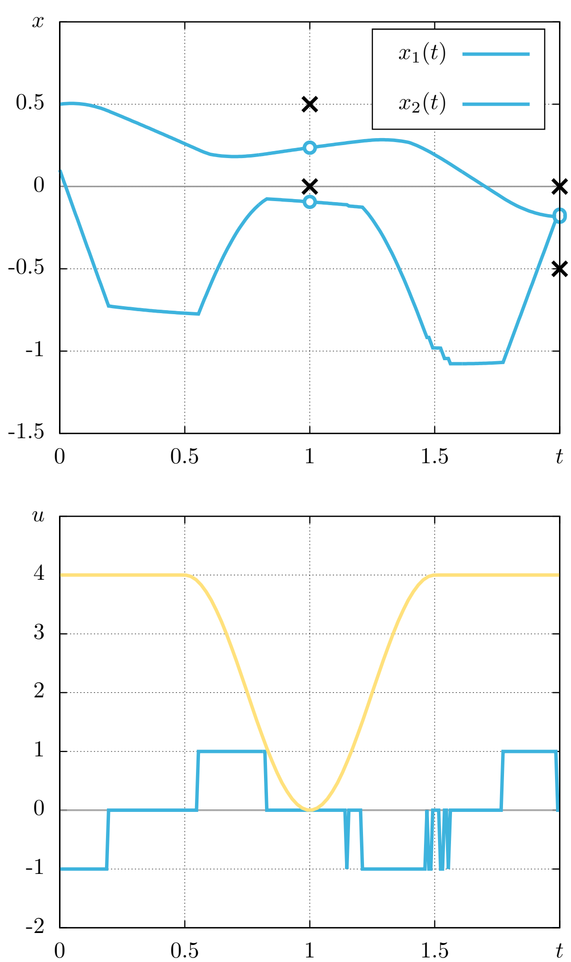

Fig. 1. Realizations of motion and control actions obtained in the first simulation.

In the second one, control actions were formed according to the "greedy" law, which performs the extremal shift to the next target while there are enough resources left. Disturbance actions were formed by the law $\mathcal{V}^e$. Under these conditions the resulting value of the quality index (5) was

$$\gamma^{(2)} = \big(|1.153|^2 + |-0.446 - 0.5|^2 + |1.440 + 0.5|^2 + \\ + |-0.248|^2\big)^{1/2} \approx 2.459 > e_1^-(x_0, \rho_0) = 0.702.$$

In the third one, control actions were formed by the law $\mathcal{U}^e$, while the disturbance actions were random. Under these conditions the resulting value of the quality index (5) was

$$\gamma^{(3)} = \big(|0.055|^2 + |-0.025 - 0.5|^2 + |-0.445 + 0.5|^2 + \\ + |-0.110|^2\big)^{1/2} \approx 0.542 < e_1^-(x_0, \rho_0) = 0.702.$$

REFERENCES

Balashov, M. V., 2002. On the p-property of compact convex sets. Mathematical Notes 71 (3–4), 295–304.

Gomoyunov, M. I., Lukoyanov, N. Y., 2015. On the stability of a procedure for solving a minimax control problem for a positional functional. Proceedings of the Steklov Institute of Mathematics 288 (1), S54–S69.

Kornev, D. V., 2012. On numerical solution of positional differential games with nonterminal payoff. Automation and Remote Control 73 (11), 1808–1821.

Krasovskii, A. N., 1987. Construction of mixed strategies on the basis of stochastic programs. Journal of Applied Mathematics and Mechanics 51 (2), 144–149.

Krasovskii, A. N., Krasovskii, N. N., 1995. Control under Lack of Information. Birkhäuser, Berlin etc.

Krasovskii, N. N., 1985. Control of a Dynamic System. Problem about the Minimum of the Guaranteed Result. Nauka, Moscow.

Krasovskii, N. N., Lukoyanov, N. Y., 1996. Problem of conflict control with hereditary information. Journal of Applied Mathematics and Mechanics 60 (6), 869–882.

Krasovskii, N. N., Reshetova, T., 1988. About programm synthesis of guaranteed control. Problemi upravleniya i teoriya informatsii 17 (6), 333–343.

Krasovskii, N. N., Tretyakov, V. E., 1966. To problem of pursuit in case of constraints on impulses of control forces. Differentsialnie Uravneniya 2 (5), 587–599.

Lokshin, M. D., 1992. Differential games with integral restrictions on the controlling actions. Differential Equations 28 (11), 1602–1610.

Lukoyanov, N. Y., 1995a. A conflict control problem under mixed restrictions on control actions. Differential Equations 31 (9), 1427–1436.

Lukoyanov, N. Y., 1995b. The problem of conflicting control with mixed constraints. Journal of Applied Mathematics and Mechanics 59 (6), 911–919.

Lukoyanov, N. Y., 1998. The problem of computing the value of a differential game for a positional functional. Journal of Applied Mathematics and Mechanics 62 (2), 177–186.

Lukoyanov, N. Y., 2001. How to compute the value of a positional differential game. Differential Equations 37 (1), 17–26.

Nikolskii, M. S., 1972. Direct method in linear differential games with common integral constraints. Differentsialnie Uravneniya 8 (6), 964–971.

Subbotina, N. N., Subbotin, A. I., 1975. Alternative for the encounter-evasion differential game with constraints on the momenta of the players' controls. Journal of Applied Mathematics and Mechanics 39 (3), 397–406.